Hypothesis test on mean and proportion

In a poultry business, type "I" egg cartons are sold. The seller informs customers of an average weight of 63g per unit. A customer who bought a carton is suspicious and measures the weight of 10 eggs with the following results:

$$\text{Weight [g]} = [60 \quad 57 \quad 65 \quad 59 \quad 65 \quad 62 \quad 63 \quad 68 \quad 60 \quad 63]$$

It is assumed that the weight of each type I egg follows a normal distribution.

a) Develop a hypothesis test to assess the seller's claim, considering the customer's concern of buying a product with a lower weight than expected. Find the critical weight that separates the rejection and non-rejection regions using a 5% significance level. Indicate the decision.

b) What is the confidence value from which the decision of the previous test would change?

c) On the other hand, it is assumed that the weight deviation of a type I egg would be less than or equal to 3g. Test this with an appropriate test at a 10% significance level. Report the p-value.

d) The customer continued measuring the entire carton (30 eggs) and found that only 18 eggs meet the seller's claim. Find a 95% confidence interval for the proportion of type I eggs that meet the seller's claim.

Solution

a) Hypothesis Test on the Average Weight

The customer is concerned about buying a product with a lower weight than expected. We set up the hypotheses:

- Null hypothesis (H_0): The average weight is at least 63 grams.

$$H_0 : \mu \geq 63 \text{ g}$$

- Alternative hypothesis (H_1): The average weight is less than 63 grams.

$$H_1 : \mu < 63 \text{ g}$$

Significance level: $\alpha = 0.05$.

Calculation of Sample Statistics

Sample size: $n = 10$
Sample mean ($\bar{x}$):

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{60+57+65+59+65+62+63+68+60+63}{10} = \frac{622}{10} = 62.2 \text{ g}$$

Sample standard deviation (s):
First, we calculate the squared deviations:

$$\begin{aligned}
(60-62.2)^2 &= (-2.2)^2 = 4.84 \\
(57-62.2)^2 &= (-5.2)^2 = 27.04 \\
(65-62.2)^2 &= (2.8)^2 = 7.84 \\
(59-62.2)^2 &= (-3.2)^2 = 10.24 \\
(65-62.2)^2 &= (2.8)^2 = 7.84 \\
(62-62.2)^2 &= (-0.2)^2 = 0.04 \\
(63-62.2)^2 &= (0.8)^2 = 0.64 \\
(68-62.2)^2 &= (5.8)^2 = 33.64 \\
(60-62.2)^2 &= (-2.2)^2 = 4.84 \\
(63-62.2)^2 &= (0.8)^2 = 0.64
\end{aligned}$$

Summing the squared deviations:

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = 4.84 + 27.04 + 7.84 + 10.24 + 7.84 + 0.04 + 0.64 + 33.64 + 4.84 + 0.64 = 97.60$$

Calculation of Sample Variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{97.60}{9} \approx 10.8444$$

The sample standard deviation is:

$$s = \sqrt{s^2} = \sqrt{10.8444} \approx 3.2946 \text{ g}$$

Calculation of the Test Statistic

We use the Student's t-distribution:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{62.2 - 63}{3.2946/\sqrt{10}} = \frac{-0.8}{1.0428} = -0.7663$$

Determination of the Critical Value

Degrees of freedom: $\nu = n - 1 = 9$

For a significance level of $\alpha = 0.05$ in a one-tailed test (left tail):

$$t_{\alpha,\nu} = t_{0.05,9} = -1.833$$

Decision Rule

Reject H_0 if $t \leq t_{\alpha,\nu}$.

Since $-0.7663 > -1.833$, we do not reject H_0.

Critical Weight

The critical weight that separates the rejection and non-rejection regions is:

$$\bar{x}_c = \mu_0 + t_{\alpha,\nu}\left(\frac{s}{\sqrt{n}}\right) = 63 + (-1.833)(1.0428) = 63 - 1.9118 = 61.0882 \text{ g}$$

Conclusion: Given that $\bar{x} = 62.2$ g > 61.0882 g, we do not reject the null hypothesis. We do not reject H_0 at the 5% significance level. The critical weight that separates the rejection and non-rejection regions is approximately 61.088 g. Since the sample mean is higher, there is not enough evidence to conclude that the eggs weigh less than what the seller claims.

b) Confidence Value to Change the Decision

We calculate the p-value associated with the test statistic:

$$p\text{-value} = P(T \leq -0.7663) \quad \text{with} \quad T \sim t(9)$$

Using tables or a statistical calculator:

$$p\text{-value} \approx 0.232$$

The associated confidence level is:

$$\text{Confidence level} = 1 - p\text{-value} = 1 - 0.232 = 0.768$$

The decision of the test would change with a confidence level below 76.8% (or a significance level above 23.2%).

c) Hypothesis Test on the Standard Deviation

We set up the hypotheses:

- Null hypothesis (H_0): The standard deviation is less than or equal to 3 grams.

$$H_0 : \sigma \leq 3 \text{ g}$$

- Alternative hypothesis (H_1): The standard deviation is greater than 3 grams.

$$H_1 : \sigma > 3 \text{ g}$$

Significance level: $\alpha = 0.10$.

Calculation of the Test Statistic

We use the chi-squared distribution:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{9 \times 10.8444}{9} = 10.8444$$

Degrees of freedom: $\nu = n - 1 = 9$

Determination of the Critical Value

For a one-tailed test (right tail) with $\alpha = 0.10$:

$$\chi^2_{\alpha,\nu} = \chi^2_{0.90,9} = 14.684$$

Decision Rule

Reject H_0 if $\chi^2 \geq \chi^2_{\alpha,\nu}$.
Since $10.8444 < 14.684$, we do not reject H_0.

Calculation of the p-value

$$p\text{-value} = P(\chi^2 \geq 10.8444) \quad \text{with} \quad \chi^2 \sim \chi^2(9)$$

Using tables or a statistical calculator:

$$p\text{-value} = 1 - P(\chi^2 \leq 10.8444) \approx 1 - 0.7028 = 0.2972$$

Conclusion: The p-value is greater than $\alpha = 0.10$, so we do not reject H_0.

We do not reject the null hypothesis at the 10% significance level. The p-value is approximately 29.72%, indicating that there is not enough evidence to conclude that the standard deviation is greater than 3 grams.

d) Confidence Interval for the Proportion

The customer measured all 30 eggs and found that 18 met the seller's claim (weight of at least 63 grams).
Sample proportion ($\hat{p}$):

$$\hat{p} = \frac{18}{30} = 0.6$$

Calculation of the 95% Confidence Interval

We use the normal approximation for proportions:

$$SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.6 \times 0.4}{30}} = \sqrt{0.008} = 0.0894$$

Critical value for $\alpha = 0.05$:

$$z_{\alpha/2} = z_{0.025} = 1.96$$

Calculating the margin of error:

$$E = z_{\alpha/2} \times SE = 1.96 \times 0.0894 = 0.1752$$

The confidence interval is:

$$\hat{p} \pm E = 0.6 \pm 0.1752 = (0.4248,\ 0.7752)$$

The 95% confidence interval for the proportion of eggs that meet the seller's claim is $(0.4248,\ 0.7752)$.